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2009 J. Phys. A: Math. Theor. 42 345204

(http://iopscience.iop.org/1751-8121/42/34/345204)

J. Phys. A: Math. Theor. **42** (2009) 345204 (27pp) doi:10.1088/1751-8113/42/34/345204

Images in quantum entanglement

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Received 11 December 2008, in final form 16 July 2009
Published 10 August 2009
Online at stacks.iop.org/JPhysA/42/345204

Abstract
A system for classifying and quantifying entanglement in spin 1/2 pure states is presented based on simple images. From the image point of view, an entangled state can be described as a linear superposition of separable object wavefunction Ψ_O plus a portion of its own inverse image. Bell states can be defined in this way: $\Psi = 1/\sqrt{2}(\Psi_O \pm \Psi_I)$. Using the method of images, the three-spin 1/2 system is discussed in some detail. This system can exhibit exclusive three-particle ν_{123} entanglement, two-particle entanglements $\nu_{12}, \nu_{13}, \nu_{23}$ and/or mixtures of all four. All four image states are orthogonal both to each other and to the object wavefunction. In general, five entanglement parameters $\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}$ and ϕ_{123} are required to define the general entangled state. In addition, it is shown that there is considerable scope for encoding numbers, at least from the classical point of view but using quantum-mechanical principles. Methods are developed for their extraction. It is shown that concurrence can be used to extract even-partite, but not odd-partite information. Additional relationships are also presented which can be helpful in the decoding process. However, in general, numerical methods are mandatory. A simple roulette method for decoding is presented and discussed. But it is shown that if the encoder chooses to use transcendental numbers for the angles defining the target function (α_1, β_1), etc, the method rapidly turns into the Devil's roulette, requiring finer and finer angular steps.

PACS numbers: 03.65.−w, 03.65.Ud, 03.65.Wj, 03.67.−a, 03.67.Dd, 03.67.Mn

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entangled states have played an important role in the development and understanding of the subject we call quantum mechanics. Key milestones include the Einstein, Podolsky and Rosen (EPR) paper (1935), Schrödinger's statement that entanglement is the *essence of quantum mechanics* (1935), Bell's inequalities (Bell 1964) and their subsequent refutation in favour of quantum mechanics (Aspect *et al* 1982). A concise history can be found in the book by Peres (1995a). More recently, entangled states have attracted interest in the field of teleportation (e.g. Bennett *et al* 1993, Brassard 1996, Zhao *et al* 2004). Indeed, entanglement is now regarded as a 'resource' with applications in quantum computing, quantum error correction and quantum cryptography (e.g. Ekert 1991, Ekert and Jozsa 1998, Bennett *et al* 1992, 1993, Shor 1995, 1996, Steane and Ibinson 2003, Knill 2005).

Clearly, the problem of classifying and quantifying the amount of entanglement in a given wavefunction is of fundamental importance. Hill and Wootters (1997) and Wootters *et al* (1998a, 1998b) have addressed this problem for the two qubit system. However, the entanglement problem in three or more qubit systems is more complex (see for example Bennett *et al* 2000, Acin *et al* 2000, Dür *et al* 2000, Miyake 2003, Sugita 2008).

In this paper, entanglement is approached from a differing viewpoint. In sections 2 and 3, a technique, which we call the method of images, is used to show that any entangled pure state function can be re-expressed in the form of a linear superposition of separable object wavefunction Ψ_O plus a portion of its own inverse image Ψ_I. In particular, a physical representation is given of a generalized two-spin 1/2 Bell state, and it is shown that the method of images is compatible with the Schmidt decomposition, at least for two-spin 1/2 particles. Contact is also made, in section 4, with the concept of concurrence developed by Wootters and co-workers (1998a, 1998b).

In sections 5–8, the problem of three-spin 1/2 particle entanglements is addressed in some detail. It is shown that exclusive three-spin entanglement ν_{123} is relatively easy to define. However, both three-spin ν_{123} and the two-spin entanglements $\nu_{12}, \nu_{13}, \nu_{23}$ can co-exist, providing additional complexity. It is also shown that W-functions of Dür *et al* (2000), fall within the framework of image-entangled states. Contact is also made with the entangled functions of Acin *et al* (2000, 2001).

In sections 9 and 10, the problem of extracting the entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}$ and ν_{123} from a given pure state is examined in some detail. While concurrence can be used to easily extract the single entanglement parameter in a two-spin 1/2 state, it enjoys only limited success in three-spin 1/2 systems. However, some new, almost projective relationships are also given, which could prove useful in the decoding process.

In section 11, some simple examples of decoding are discussed. These examples pave the way for the roulette method discussed in section 12. In essence, the roulette method can be described as a brute force technique for decoding entanglement parameters.

Finally, in sections 13 and 14, some comments are made concerning the classification of image-entangled states. In particular, it is shown that for n-spin 1/2 systems, the number of entanglement parameters (real, complex) is given by $(N_\nu = (2^n - (n+1))$, $N_{\nu\phi} = (2^{n+1} - (3n+2))$, respectively. These numbers are compatible with those of Linden and Popescu (1997) obtained by another route. As n is increased therefore, the decoding-problem escalates rapidly in complexity. Nonetheless, for all n-spin 1/2 particle systems, the simplest is that of exclusive n-particle entanglement $\nu_{123,\ldots,n}$. Indeed, it is these entangled wavefunctions, in their maximally entangled form ($\nu_{123..n} = \pm 1$), which give rise to the so-called Schrödinger 'Cat-states' (e.g. Bennett *et al* 2000, Leibfried *et al* 2005).

2. Two-spin 1/2 particles: image entanglement

Consider the general two-particle pure state:

$$|\Psi\rangle = a|+\rangle_1|+\rangle_2 + b|+\rangle_1|-\rangle_2 + c|-\rangle_1|+\rangle_2 + d|-\rangle_1|-\rangle_2, \tag{1}$$

where (i) $|\pm\rangle$ is a shorthand notation for the two Zeeman states $\left|\pm\frac{1}{2}\right\rangle$ and (ii)

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1. \tag{2}$$

Within the method of images, we shall assume, at least initially, that the coefficients a—d, etc, are real. But this is not a limitation for two-spin 1/2 systems, since any phases can be incorporated into the definitions of the basis sets (see Peres 1995a and appendix A).

We now present an image technique which is central to our discussion. Any two-spin 1/2 entangled state can be written in the form:

$$|\Psi\rangle = {}^1\!/\!\sqrt{1+\nu^2}(|\Psi\rangle_O + \nu|\Psi\rangle_I), \tag{3}$$

where (i)

$$|\Psi\rangle_O = {}^1\!/\!\sqrt{1+\nu^2}((\alpha_1|+\rangle_1 - \beta_1|-\rangle_1) \otimes (\alpha_2|+\rangle_2 - \beta_2|-\rangle_2))$$

and (4)

$$|\Psi\rangle_I = \nu\,{}^1\!/\!\sqrt{1+\nu^2}((\beta_1|+\rangle_1 + \alpha_1|-\rangle_1) \otimes (\beta_2|+\rangle_2 + \alpha_2|-\rangle_2)).$$

Here we have used the notation $|\Psi\rangle_O$ and $|\Psi\rangle_I$ to signify object and image wavefunctions, respectively, for reasons that will become apparent in section 3. Note that (i) taken separately $|\Psi\rangle_O$ and $|\Psi\rangle_I$ are separable and (ii) they are orthogonal to each other: $\langle\Psi_I|\Psi_0\rangle = 0$ and (iii) they are phase locked. Thus the states defined by equation (3) are pure state functions, defined by essentially three parameters: ($\alpha_1, \alpha_2,$ and ν), but with possible extra phases locked away in the basis states (see appendix A). Indeed, most authors would assert that equation (3) is obvious. Any pure state can be reduced to just two terms, by an appropriate Schmidt decomposition (see Peres 1995a, Acin *et al* 2000 for a very short proof). But, while the image approach can be easily extended to three- or more spin 1/2 particles, by way of contrast, this does not apply to the Schmidt decomposition (e.g. Peres 1995a, 1995b). Finally, note that we have chosen to place the negative signs in the object wavefunction, rather than the image wavefunction. This is unimportant but leads to some advantages in multi-spin 1/2 systems.

Next we equate the coefficients of equations (1) and (3). In matrix form we find

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathsf{E}_2 \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix} = \frac{1}{\sqrt{1+\nu^2}} \begin{bmatrix} 1 & 0 & 0 & \nu \\ 0 & -1 & \nu & 0 \\ 0 & \nu & -1 & 0 \\ \nu & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix}, \tag{5}$$

where E_2 can be described as an entanglement matrix. Note that $\mathsf{E}_2^{-1} \neq \mathsf{E}_2^{\dagger}$, thus the entanglement matrix E_2 is not unitary in the usual sense. Nevertheless, equation (5) can easily be used to show that the normalization condition of equations (2) and (4) hold, regardless of the value of the entanglement admixture ν. This, of course, is not entirely unexpected given that $|\Psi\rangle_O$ and $|\Psi\rangle_I$ are 'doubly' orthogonal to each other.

Next, we observe that equation (5) possesses the inverse:

$$\begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{bmatrix} = \mathsf{E}_2^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{\sqrt{1+\nu^2}}{1-\nu^2} \begin{bmatrix} 1 & 0 & 0 & -\nu \\ 0 & -1 & -\nu & 0 \\ 0 & -\nu & -1 & 0 \\ -\nu & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}. \tag{6}$$

Consequently

$$\begin{aligned} \alpha_1\alpha_2 &= +\eta(a - \nu d) \\ \alpha_1\beta_2 &= -\eta(b + \nu c) \\ \beta_1\alpha_2 &= -\eta(c + \nu b) \qquad \eta = \tfrac{\sqrt{1+\nu^2}}{1-\nu^2} \\ \beta_1\beta_2 &= +\eta(d - \nu a) \end{aligned} \tag{7}$$

which reveals:

$$\frac{\alpha_1}{\beta_1} = -\frac{(a - \nu d)}{(c + \nu b)} = -\frac{(b + \nu c)}{(d - \nu a)},$$
$$\frac{\alpha_2}{\beta_2} = -\frac{(a - \nu d)}{(b + \nu c)} = -\frac{(c + \nu b)}{(d - \nu a)}. \tag{8}$$

Either of these two equations can be used to determine the value of the admixture ν. We find

$$\nu = \frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{\gamma^2 - 4}, \qquad \text{where} \quad \gamma = \frac{1}{(ad - bc)}. \tag{9}$$

Here the plus sign is appropriate when $\nu \geqslant 0$, and the negative sign when $\nu < 0$. Given the entanglement parameter ν therefore, we can compute the ratios α_1/β_1 and α_2/β_2 from equation (8). Thus, starting from the four coefficients *a*–*d*, we can re-construct $|\Psi\rangle$ in the form of equations (3) and (4).

We are now in a position to make contact with the Bell states. If $\nu = 0$, $|\Psi\rangle$ is fully separable, while if $\nu = \pm 1$, $|\Psi\rangle$ is fully entangled. To see that this is so consider the situation when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 0$. In this case, equation (3) reduces to

$$|\Psi\rangle = \frac{1}{\sqrt{1+\nu^2}}(|+\rangle_1|+\rangle_2 + \nu|-\rangle_1|-\rangle_2) \tag{10}$$

For $\nu = \pm 1$ therefore, we find two maximally entangled Bell states. Similarly, for $\alpha_1 = \beta_2 = 1$, $\alpha_2 = \beta_1 = 0$:

$$|\Psi\rangle = \frac{1}{\sqrt{1+\nu^2}}(\nu|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2). \tag{11}$$

From the above discussion it is clear that the admixture $-1 \leqslant \nu \leqslant 1$ provides us with a measure of the degree of entanglement. Of course, larger values of ν are allowed. But for $|\nu| > 1$, the roles of $|\Psi\rangle_O$ and $|\Psi\rangle_I$ simply reverse, with $|\Psi\rangle_O$ now being the partially entangled state of $|\Psi\rangle_I$.

In summary, equation (3) can be viewed as an image entangled state. As we shall see its generalization to three or more spins is straightforward but non-trivial. In the following section, a physical interpretation of the image entangled state for two-spin 1/2 particles is briefly discussed.

3. Generalized Bell entanglement

Consider a spin 1/2 particle in the spin-up state $|\Psi\rangle = \left|\frac{1}{2}\right\rangle$. If we rotate the axes through the Euler angles α', β', γ', the new wavefunction is given by

$$|\Psi\rangle' = \mathcal{D}^{1/2}(\alpha'\beta'\gamma')\left|\tfrac{1}{2}\right\rangle, \tag{12}$$

where the $\mathcal{D}^{1/2}(\alpha'\beta'\gamma')$ are the well-known *SU*(2) rotation operators for a spin 1/2 particle (e.g. Edmonds 1960, Varshalovich *et al* 1989). Note that we have used the symbols $(\alpha'\beta'\gamma')$ to distinguish between the rotation angles and the coefficients of the wavefunctions $\alpha_1, \alpha_2, \beta_1$ and β_2 discussed above. But, as we shall see below, they are closely connected.

First, we rotate the particle through the Euler angles α', β', γ', with respect to fixed axes. Here the appropriate rotation operator to use is given by $\mathcal{D}^{1/2}(-\alpha' - \beta' - \gamma')$ (Wolf 1969). For simplicity, we shall set the Euler angles $(\alpha' = \gamma' = 0)$, or alternately absorb them into the basis states (see appendix A). This guarantees that the coefficients are real. In this case, equation (12) reduces to

$$|\Psi\rangle' = \sum_m \mathrm{d}^{1/2}_{m,1/2}(-\beta')|\mathrm{m}\rangle. \tag{13}$$

In the matrix form therefore

$$\begin{aligned}|\Psi\rangle' &= \begin{bmatrix}\cos\left(\frac{\beta'}{2}\right) & -\sin\left(\frac{\beta'}{2}\right)\\ +\sin\left(\frac{\beta'}{2}\right) & \cos\left(\frac{\beta'}{2}\right)\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\left(\frac{\beta'}{2}\right)\\ \sin\left(\frac{\beta'}{2}\right)\end{pmatrix}\\ &= \cos\left(\frac{\beta'}{2}\right)|+\rangle + \sin\left(\frac{\beta'}{2}\right)|-\rangle.\end{aligned} \tag{14}$$

Thus starting from a particle in the spin-up state, we can generate any given state $\left(\alpha_1\left|\frac{1}{2}\right\rangle_1 + \beta_1\left|-\frac{1}{2}\right\rangle_1\right)$, where $\alpha_1 = \cos\left(\frac{\beta'}{2}\right)$ and $\beta_1 = \sin\left(\frac{\beta'}{2}\right)$. Note that the normalization condition is satisfied automatically.

We are now in a position to give a physical interpretation of the image entangled state discussed in section 2. Instead of rotating by β' in equation (14), we rotate by $\beta' + \pi$:

$$\begin{aligned}|\Psi\rangle'' &= \begin{bmatrix}\cos\left(\frac{\beta'+\pi}{2}\right) & -\sin\left(\frac{\beta'+\pi}{2}\right)\\ +\sin\left(\frac{\beta'+\pi}{2}\right) & \cos\left(\frac{\beta'+\pi}{2}\right)\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\sin\left(\frac{\beta'}{2}\right)\\ +\cos\left(\frac{\beta'}{2}\right)\end{pmatrix}\\ &= -\sin\left(\frac{\beta'}{2}\right)|+\rangle + \cos\left(\frac{\beta'}{2}\right)|-\rangle.\end{aligned} \tag{15}$$

Here the values of α_1, β_1 have been swapped in equations (14) and (15) along with a sign change. So we interpret $|\Psi\rangle$ of equation (3) as the sum of the separable object wavefunction $|\Psi\rangle_O$ plus its inverse image $|\Psi\rangle_I$. In general, therefore, a generalized Bell state can be expressed in the form:

$$|\Psi\rangle = \phi_{1a}\otimes\phi_{2a} \pm \phi_{1b}\otimes\phi_{2b}$$

where

$$\begin{aligned}&\phi_{1a} = \mathcal{D}^{1/2}(-\alpha' - \beta' - \gamma')\left|\tfrac{1}{2}\right\rangle_1; \qquad \phi_{2a} = \mathcal{D}^{1/2}(-\alpha'' - \beta'' - \gamma'')\left|\tfrac{1}{2}\right\rangle_2\\ &\phi_{1b} = \mathcal{D}^{1/2}(-\alpha' - (\beta'+\pi) - \gamma')\left|\tfrac{1}{2}\right\rangle_1; \qquad \phi_{2b} = \mathcal{D}^{1/2}(-\alpha'' - (\beta''+\pi) - \gamma'')\left|\tfrac{1}{2}\right\rangle_2\end{aligned} \tag{16}$$

A pictorial representation can be seen in figure 1.

Note that (i) similar conclusions can also be reached if the inverse state is obtained by rotating through 3π radians, for spin 1/2 systems and (ii) the admixture or entanglement parameter υ is invariant for local operations (rotations) are carried out on the basis states.

In summary, an entangled state is one which possesses at least a portion of its own inverse-image.

4. Concurrence and entangled bi-partite states

The problem of entanglement in both pure and mixed states has been considered by several authors, notably Bennett *et al* (1996) and Wootters *et al* (1998a, 1998b). In particular, Bennett *et al* (1996) have shown that entanglement $E(\Psi)$ for a pure bi-partite states, can be defined as the von Neumann entropy (or equivalently the Shannon entropy) of either the reduced density matrices ρ_A or ρ_B of a bi-partite AB system (Bennett *et al* 1996). However, for pure two qubit states Wootters has introduced 'concurrence' as one of the measures of entanglement.

Using a slight adaption of equation (7) of Wootters (1998b), we define the concurrence $C(\Psi) = \langle\Psi_\pi|\Psi\rangle$, where

$$|\Psi_\pi\rangle = I_{12}(\pi)|\Psi\rangle = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}_1 \otimes \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}_2 |\Psi\rangle, \tag{17}$$

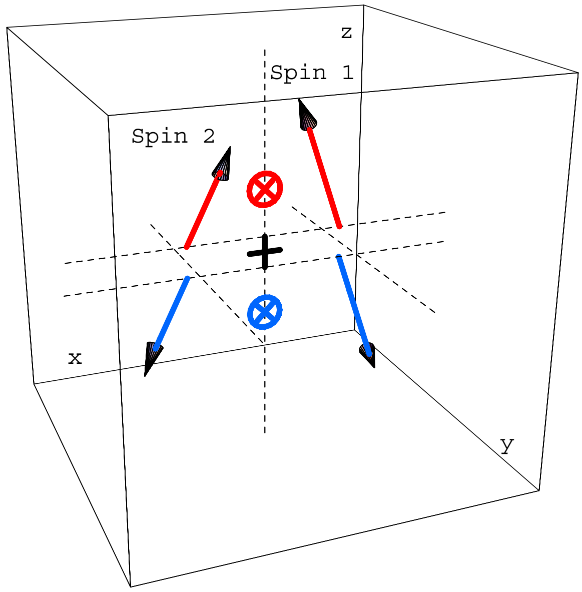

Figure 1. A pictorial representation of a generalized Bell state.

i.e. a rotation of $\beta = \pi$ on both particles. Thus the concurrence $C(\Psi)$ of the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{1+\nu^2}}\left\{\begin{pmatrix}\cos\left(\frac{\beta'}{2}\right)\\ -\sin\left(\frac{\beta'}{2}\right)\end{pmatrix}_1 \otimes \begin{pmatrix}\cos\left(\frac{\beta''}{2}\right)\\ -\sin\left(\frac{\beta''}{2}\right)\end{pmatrix}_2 + \nu \begin{pmatrix}\sin\left(\frac{\beta'}{2}\right)\\ \cos\left(\frac{\beta'}{2}\right)\end{pmatrix}_1 \otimes \begin{pmatrix}\sin\left(\frac{\beta''}{2}\right)\\ \cos\left(\frac{\beta''}{2}\right)\end{pmatrix}_2\right\}$$

$$= a|{+}{+}\rangle + b|{+}{-}\rangle + c|{-}{+}\rangle + d|{-}{-}\rangle \qquad (18)$$

is given by

$$C(\Psi) = \langle\Psi_\pi|\Psi\rangle = 2(ad - bc) = \frac{2\nu}{1+\nu^2}. \qquad (19)$$

Given the four coefficients $\{a, b, c, d\}$ therefore, the entanglement parameter ν can be easily projected out, without the need to establish the four coefficients $\alpha_1, \beta_1, \alpha_2, \beta_2$. This constitutes a great saving in effort. A plot of $C(\Psi)$ can be seen in figure 2. Note that we have dropped the modulus sign of Wootters. The latter places the concurrence of a pure state, and the entropic entanglement $0 \leqslant E(\Psi) \leqslant 1$, on an equal footing (i.e. positive semi-definite). However, we choose to use concurrence to determine both the magnitude and sign of entanglement parameters, in $n = 2,3$, etc, spin 1/2 systems.

This completes our discussion of the bi-partite problem. We turn now to a discussion of entanglement in three-spin 1/2 systems.

5. Exclusive three-particle entanglement: GHZ states

Once again we shall assume, for the most part, that the coefficients of the three-spin 1/2 are real. Thus at first sight, this assumption would appear to limit the number of entangled states that can be examined (see for example Dür *et al* 2000, Miyake 2003, Sugita 2008). However the limitations are not as severe as might at first be anticipated, since most phases (bar one) can be absorbed into the basis sets (see section 13 and appendix D).

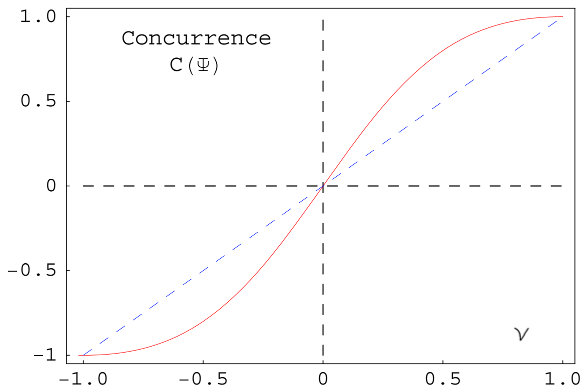

Figure 2. Modified concurrence $C(\Psi)$ as a function of ν.

First, let us assume that we are faced with exclusive three-particle entanglement. In this case we make the ansatz:

$$|\Psi\rangle = \frac{1}{\sqrt{1+\nu^2}}\begin{bmatrix}(\alpha_1|+\rangle_1 - \beta_1|-\rangle_1)\otimes(\alpha_2|+\rangle_2 - \beta_2|-\rangle_2)\otimes(\alpha_3|+\rangle_3 - \beta_3|-\rangle_3)\\ +\nu(\beta_1|+\rangle_1 + \alpha_1|-\rangle_1)\otimes(\beta_2|+\rangle_2 + \alpha_2|-\rangle_2)\otimes(\beta_3|+\rangle_3 + \alpha_3|-\rangle_3)\end{bmatrix}, \tag{20}$$

where once again certain phases have been absorbed into the basis states (see appendix B).

On equating coefficients therefore we find

$$\begin{aligned}
a &= \frac{1}{\eta}(\alpha_1\alpha_2\alpha_3 + \nu\beta_1\beta_2\beta_3), & e &= \frac{1}{\eta}(-\beta_1\alpha_2\alpha_3 + \nu\alpha_1\beta_2\beta_3),\\
b &= \frac{1}{\eta}(-\alpha_1\alpha_2\beta_3 + \nu\beta_1\beta_2\alpha_3), & f &= \frac{1}{\eta}(\beta_1\alpha_2\beta_3 + \nu\alpha_1\beta_2\alpha_3),\\
c &= \frac{1}{\eta}(-\alpha_1\beta_2\alpha_3 + \nu\beta_1\alpha_2\beta_3), & g &= \frac{1}{\eta}(\beta_1\beta_2\alpha_3 + \nu\alpha_1\alpha_2\beta_3),\\
d &= \frac{1}{\eta}(\alpha_1\beta_2\beta_3 + \nu\beta_1\alpha_2\alpha_3), & h &= \frac{1}{\eta}(-\beta_1\beta_2\beta_3 + \nu\alpha_1\alpha_2\alpha_3),
\end{aligned}$$
$$\eta = \sqrt{1+\nu^2}. \tag{21}$$

As expected, it is easy to show that the normalization condition $a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2=1$ holds, regardless of the value of ν.

Next, we rewrite equation (21) in the matrix form:

$$\begin{bmatrix}a\\b\\c\\d\\e\\f\\g\\h\end{bmatrix} = \mathsf{E}_3\begin{bmatrix}\alpha_1\alpha_2\alpha_3\\ \alpha_1\alpha_2\beta_3\\ \alpha_1\beta_2\alpha_3\\ \alpha_1\beta_2\beta_3\\ \beta_1\alpha_2\alpha_3\\ \beta_1\alpha_2\beta_3\\ \beta_1\beta_2\alpha_3\\ \beta_1\beta_2\beta_3\end{bmatrix}, \tag{22}$$

where the entanglement matrix E_3 is given by the Hermitean 8×8 matrix:

$$\mathsf{E}_3 = \frac{1}{\sqrt{1+\nu^2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \nu \\ 0 & -1 & 0 & 0 & 0 & 0 & \nu & 0 \\ 0 & 0 & -1 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 1 & \nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu & -1 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 & 0 & 1 & 0 & 0 \\ 0 & \nu & 0 & 0 & 0 & 0 & 1 & 0 \\ \nu & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}. \tag{23}$$

It is easily shown that this matrix is its own inverse $\mathsf{E}_3^{-1} = \mathsf{E}_3^{\dagger} = \mathsf{E}_3$. In this case therefore the E_3 matrix is unitary, a property which holds for all odd number of spin systems $\mathsf{E}_5, \mathsf{E}_7, \ldots$. Thus the inverse of equation (23) is given by

$$\begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{bmatrix} = \mathsf{E}_3 \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}. \tag{24}$$

Once again the normalization condition:

$$\begin{pmatrix} \alpha_1^2\alpha_2^2\alpha_3^2 + \alpha_1^2\alpha_2^2\beta_3^2 + \alpha_1^2\beta_2^2\alpha_3^2 + \alpha_1^2\beta_2^2\beta_3^2 + \\ \beta_1^2\alpha_2^2\alpha_3^2 + \beta_1^2\alpha_2^2\beta_3^2 + \beta_1^2\beta_2^2\alpha_3^2 + \beta_1^2\beta_2^2\beta_3^2 \end{pmatrix} = 1 \tag{25}$$

is satisfied, independent of the admixture ν.

To make further progress, we now determine ν, etc, starting from the coefficients a–h. From equation (24) we find

$$\begin{aligned} \alpha_1\alpha_2\alpha_3 &= \frac{1}{\eta}(+a + \nu h) & \beta_1\alpha_2\alpha_3 &= \frac{1}{\eta}(-e + \nu d) \\ \alpha_1\alpha_2\beta_3 &= \frac{1}{\eta}(-b + \nu g) & \beta_1\alpha_2\beta_3 &= \frac{1}{\eta}(+f + \nu c) \\ \alpha_1\beta_2\alpha_3 &= \frac{1}{\eta}(-c + \nu f) & \beta_1\beta_2\alpha_3 &= \frac{1}{\eta}(+g + \nu b) \\ \alpha_1\beta_2\beta_3 &= \frac{1}{\eta}(+d + \nu e) & \beta_1\beta_2\beta_3 &= \frac{1}{\eta}(-h + \nu a). \end{aligned} \tag{26}$$

Clearly a great deal of correlation exists between the eight coefficients a–h. From four of these equations it is easy to show that

$$\frac{\alpha_3}{\beta_3} = \frac{a + \nu h}{-b + \nu g} = \frac{g + \nu b}{-h + \nu a}. \tag{27}$$

This equation can be used to obtain an expression for obtain an expression for ν:

$$\nu = \tfrac{1}{2}\gamma \pm \tfrac{1}{2}\sqrt{\gamma^2 + 4}, \tag{28}$$

where

$$\gamma = \frac{a^2 + b^2 - g^2 - h^2}{bg - ah}. \tag{29}$$

Here the plus sign in equation (28) is appropriate if ν is negative, and negative if ν is positive. Given ν the ratios α_3/β_3, etc, can easily be obtained from equation (27). Note that there are several routes to the value for ν. But because of the correlations between the coefficients *a*–*h*, they all lead to the same result.

In the following section, it is shown that three-spin 1/2 systems can exhibit far more complex behaviour than that implied by equation (20).

6. Combined three- and two-particle entanglement

In general, a three-spin 1/2 system can exhibit *both* two- and three-particle entanglements. For example, consider the state

$$|\Psi\rangle = \frac{1}{\eta}\begin{bmatrix} (\alpha_1|+\rangle_1 - \beta_1|-\rangle_1)\otimes(\alpha_2|+\rangle_2 - \beta_2|-\rangle_2)\otimes(\alpha_3|+\rangle_3 - \beta_3|-\rangle_3) \\ +\nu_{23}(\alpha_1|+\rangle_1 - \beta_1|-\rangle_1)\otimes(\beta_2|+\rangle_2 + \alpha_2|-\rangle_2)\otimes(\beta_3|+\rangle_3 + \alpha_3|-\rangle_3) \\ +\nu_{123}(\beta_1|+\rangle_1 + \alpha_1|-\rangle_1)\otimes(\beta_2|+\rangle_2 + \alpha_2|-\rangle_2)\otimes(\beta_3|+\rangle_3 + \alpha_3|-\rangle_3) \end{bmatrix}$$

$$\eta = \sqrt{1 + \nu_{23}^2 + \nu_{123}^2}. \tag{30}$$

Here there is partial entanglement between particles {2, 3} (ν_{23}) and partial three-body entanglement (ν_{123}) between particles {1, 2, 3}. Note that all three components in equation (30) are orthogonal to each other. Our problem therefore is to determine both ν_{23} and ν_{123}.

On equating coefficients we find

$$\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix} 1 & 0 & 0 & \nu_{23} & 0 & 0 & 0 & \nu_{123} \\ 0 & -1 & \nu_{23} & 0 & 0 & 0 & \nu_{123} & 0 \\ 0 & \nu_{23} & -1 & 0 & 0 & \nu_{123} & 0 & 0 \\ \nu_{23} & 0 & 0 & 1 & \nu_{123} & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_{123} & -1 & 0 & 0 & -\nu_{23} \\ 0 & 0 & \nu_{123} & 0 & 0 & 1 & -\nu_{23} & 0 \\ 0 & \nu_{123} & 0 & 0 & 0 & -\nu_{23} & 1 & 0 \\ \nu_{123} & 0 & 0 & 0 & -\nu_{23} & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{bmatrix}$$

$$= \mathbf{U}\begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{bmatrix} \tag{31}$$

which possesses the inverse

$$\begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{bmatrix} = \kappa\begin{bmatrix} \chi_1 & 0 & 0 & \chi_2 & \chi_3 & 0 & 0 & \chi_4 \\ 0 & -\chi_1 & \chi_2 & 0 & 0 & -\chi_3 & \chi_4 & 0 \\ 0 & \chi_2 & -\chi_1 & 0 & 0 & \chi_4 & -\chi_3 & 0 \\ \chi_2 & 0 & 0 & \chi_1 & \chi_4 & 0 & 0 & \chi_3 \\ \chi_3 & 0 & 0 & \chi_4 & -\chi_1 & 0 & 0 & \chi_2 \\ 0 & -\chi_3 & \chi_4 & 0 & 0 & \chi_1 & \chi_2 & 0 \\ 0 & \chi_4 & -\chi_3 & 0 & 0 & \chi_2 & \chi_1 & 0 \\ \chi_4 & 0 & 0 & \chi_3 & \chi_2 & 0 & 0 & -\chi_1 \end{bmatrix}\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{bmatrix}, \tag{32}$$

where

$$\begin{aligned}
&\chi_1 = \left(1 - \nu_{23}^2 + \nu_{123}^2\right), \qquad \chi_2 = -\nu_{23}\left(1 - \nu_{23}^2 - \nu_{123}^2\right)\\
&\chi_3 = -2\nu_{23}\nu_{123}, \qquad \chi_4 = \nu_{123}\left(1 + \nu_{23}^2 + \nu_{123}^2\right)\\
&\kappa = \frac{\sqrt{\left(1 + \nu_{23}^2 + \nu_{123}^2\right)}}{\left((1-\nu_{23})^2 + \nu_{123}^2\right) \times \left((1+\nu_{23})^2 + \nu_{123}^2\right)} = \frac{\eta}{\left(\eta^4 - 4\nu_{23}^2\right)}.
\end{aligned} \tag{33}$$

With the aid of this transformation therefore we find

$$\begin{aligned}
\alpha_1\alpha_2\alpha_3 &= \kappa(+a\chi_1 + d\chi_2 + e\chi_3 + h\chi_4), & \beta_1\alpha_2\alpha_3 &= \kappa(+a\chi_3 + d\chi_4 - e\chi_1 + h\chi_2),\\
\alpha_1\alpha_2\beta_3 &= \kappa(-b\chi_1 + c\chi_2 - f\chi_3 + g\chi_4), & \beta_1\alpha_2\beta_3 &= \kappa(-b\chi_3 + c\chi_4 + f\chi_1 + g\chi_2),\\
\alpha_1\beta_2\alpha_3 &= \kappa(+b\chi_2 - c\chi_1 + f\chi_4 - g\chi_3), & \beta_1\beta_2\alpha_3 &= \kappa(+b\chi_4 - c\chi_3 + f\chi_2 + g\chi_1),\\
\alpha_1\beta_2\beta_3 &= \kappa(+a\chi_2 + d\chi_1 + e\chi_4 + h\chi_3), & \beta_1\beta_2\beta_3 &= \kappa(+a\chi_4 + d\chi_3 + e\chi_2 - h\chi_1).
\end{aligned} \tag{34}$$

Consequently

$$\begin{aligned}
\frac{\alpha_1}{\beta_1} &= \frac{(a\chi_1 + d\chi_2 + e\chi_3 + h\chi_4)}{(a\chi_3 + d\chi_4 - e\chi_1 - h\chi_2)} = \frac{(-b\chi_1 + c\chi_2 - f\chi_3 + g\chi_4)}{(-b\chi_3 + c\chi_4 + f\chi_1 + g\chi_2)}\\
&= \frac{(b\chi_2 - c\chi_1 + f\chi_4 - g\chi_3)}{(b\chi_4 - c\chi_3 + f\chi_2 + g\chi_1)} = \frac{(a\chi_2 + d\chi_1 + e\chi_4 + h\chi_3)}{(a\chi_4 + d\chi_3 + e\chi_2 - h\chi_1)}
\end{aligned} \tag{35}$$

with similar expressions for α_2/β_2 and α_3/β_3.

From equation (35) we can pair off any two of the four values for α_1/β_1 and hence obtain a solution for ν_{23} and ν_{123}. In practice, this can be achieved using programs such as Mathematica with the *FindRoots* routine. But the procedure is tedious, given that starting values have to be supplied.

7. General three-particle entanglement

Finally, we address the general three-spin 1/2 entanglement problem: namely the co-existence of ν_{12}, ν_{13}, ν_{23} and ν_{123} entanglement. This time the wavefunction takes the form:

$$|\Psi\rangle = \frac{1}{\eta}\begin{bmatrix}
(\alpha_1|+\rangle_1 - \beta_1|-\rangle_1) \otimes (\alpha_2|+\rangle_2 - \beta_2|-\rangle_2) \otimes (\alpha_3|+\rangle_3 - \beta_3|-\rangle_3)\\
+\nu_{23}(\alpha_1|+\rangle_1 - \beta_1|-\rangle_1) \otimes (\beta_2|+\rangle_2 + \alpha_2|-\rangle_2) \otimes (\beta_3|+\rangle_3 + \alpha_3|-\rangle_3)\\
+\nu_{12}(\beta_1|+\rangle_1 + \alpha_1|-\rangle_1) \otimes (\beta_2|+\rangle_2 + \alpha_2|-\rangle_2) \otimes (\alpha_3|+\rangle_3 - \beta_3|-\rangle_3)\\
+\nu_{13}(\beta_1|+\rangle_1 + \alpha_1|-\rangle_1) \otimes (\alpha_2|+\rangle_2 - \beta_2|-\rangle_2) \otimes (\beta_3|+\rangle_3 + \alpha_3|-\rangle_3)\\
+\nu_{123}(\beta_1|+\rangle_1 + \alpha_1|-\rangle_1) \otimes (\beta_2|+\rangle_2 + \alpha_2|-\rangle_2) \otimes (\beta_3|+\rangle_3 + \alpha_3|-\rangle_3)
\end{bmatrix} \tag{36}$$

$$\eta = \sqrt{1 + \nu_{23}^2 + \nu_{12}^2 + \nu_{13}^2 + \nu_{123}^2}.$$

On equating coefficients we find that

$$\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{bmatrix} = \frac{1}{\eta}\begin{bmatrix}
1 & 0 & 0 & \nu_{23} & 0 & \nu_{13} & \nu_{12} & \nu_{123}\\
0 & -1 & \nu_{23} & 0 & \nu_{13} & 0 & \nu_{123} & -\nu_{12}\\
0 & \nu_{23} & -1 & 0 & \nu_{12} & \nu_{123} & 0 & -\nu_{13}\\
\nu_{23} & 0 & 0 & 1 & \nu_{123} & -\nu_{12} & -\nu_{13} & 0\\
0 & \nu_{13} & \nu_{12} & \nu_{123} & -1 & 0 & 0 & -\nu_{23}\\
\nu_{13} & 0 & \nu_{123} & -\nu_{12} & 0 & 1 & -\nu_{23} & 0\\
\nu_{12} & \nu_{123} & 0 & -\nu_{13} & 0 & -\nu_{23} & 1 & 0\\
\nu_{123} & -\nu_{12} & -\nu_{13} & 0 & -\nu_{23} & 0 & 0 & -1
\end{bmatrix}\begin{bmatrix} \alpha_1\alpha_2\alpha_3\\ \alpha_1\alpha_2\beta_3\\ \alpha_1\beta_2\alpha_3\\ \alpha_1\beta_2\beta_3\\ \beta_1\alpha_2\alpha_3\\ \beta_1\alpha_2\beta_3\\ \beta_1\beta_2\alpha_3\\ \beta_1\beta_2\beta_3 \end{bmatrix}. \tag{37}$$

This deceptively simple transformation matrix can be inverted algebraically. But the resulting matrix elements are very complicated with no zeros. In practice, it may be possible to make progress using partitioned matrix theory, since the 8×8 matrix of equation (37) can be broken down into smaller 4×4 matrices. Nonetheless, general progress can be made as follows.

First, we re-define the inverse transformation:

$$\begin{bmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{bmatrix} = \mathbf{U}^{-1} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \mathbf{U}^{-1} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix}, \tag{38}$$

where, for simplicity, we have redefined the coefficients $(a - h)$ as $(a_1 - a_8)$. Thus,

$$\begin{aligned} \alpha_1\alpha_2\alpha_3 &= \sum_j (\mathbf{U}^{-1})_{1j} a_j \\ \alpha_1\alpha_2\beta_3 &= \sum_j (\mathbf{U}^{-1})_{2j} a_j \\ &\ldots \\ \beta_1\beta_2\beta_3 &= \sum_j (\mathbf{U}^{-1})_{8j} a_j. \end{aligned} \tag{39}$$

Once again, we insist that the four solutions for say the α_3/β_3, etc, must, of course, be identical:

$$\frac{\alpha_3}{\beta_3} = \frac{\sum_j (U^{-1})_{1j} a_j}{\sum_j (U^{-1})_{2j} a_j} = \frac{\sum_j (U^{-1})_{3j} a_j}{\sum_j (U^{-1})_{4j} a_j} = \frac{\sum_j (U^{-1})_{5j} a_j}{\sum_j (U^{-1})_{6j} a_j} = \frac{\sum_j (U^{-1})_{7j} a_j}{\sum_j (U^{-1})_{8j} a_j}. \tag{40}$$

Likewise for α_2/β_2, and α_1/β_1. From such equations we can determine the coefficients $\nu_{12}, \nu_{13}, \nu_{23}$ and ν_{123} and hence obtain a full solution for the multiply entangled three-particle spin $1/2$ system.

In summary, the three-spin $1/2$ image entangled systems can exhibit very complex behaviour, involving the co-existence of three-particle ν_{123} and two-particle entanglements ν_{12}, ν_{13} and ν_{23}. These conclusions find a natural resonance with the comments of Bennett *et al* (2000). The latter note that there are different classes of entanglement, and stress that the maximally entangled tri-partite or $|\text{GHZ}\rangle$ Cat-states (exclusive three-particle entanglement) cannot be expressed in terms of bi-partite (EPR) entangled-states. These ideas have been extended by Dür *et al* (2000), Miyake (2003) and Sugita (2008), using group theoretical methods. In particular, these authors have shown that in addition to $|\text{GHZ}\rangle$ and $|\text{EPR}\rangle$ states there are $|W\rangle$ states. These are discussed in the following section.

8. *W*-entangled states

Dür *et al* (2000) identified a new class of $|W\rangle$ states that are orthogonal to the $|\text{GHZ}\rangle$ states (see also Schlienz and Mahler 1996). One such state takes the form

$$\begin{aligned} |W\rangle &= \tfrac{1}{\sqrt{3}}\{|+\rangle_1|+\rangle_2|-\rangle_3 + |+\rangle_1|-\rangle_2|+\rangle_3 + |-\rangle_1|+\rangle_2|+\rangle_3\} \\ &\left(\equiv \tfrac{1}{\sqrt{3}}\{|001\rangle + |010\rangle + |100\rangle\}\right) \end{aligned} \tag{41}$$

(see equation (2) of Dür *et al* 2000). This function falls within the general framework of equation (36). If we set $\alpha_1 = \alpha_2 = 1, \beta_3 = -1$ and $\nu_{12} = 0, \nu_{13} = \nu_{23} = -1, \nu_{123} = 0,$

we find the $|W\rangle$ function of equation (41). Likewise, if we place $\alpha_1 = 1$, $\beta_2 = \beta_3 = 1$ and $\nu_{23} = 0$, $\nu_{12} = \nu_{13} = -1$, $\nu_{123} = 0$ we find

$$|W\rangle_I = \frac{1}{\sqrt{3}}\{|-\rangle_1|-\rangle_2|+\rangle_3 + |-\rangle_1|+\rangle_2|-\rangle_3 + |+\rangle_1|-\rangle_2|-\rangle_3\} \\ \left(\equiv \frac{1}{\sqrt{3}}\{|110\rangle + |101\rangle + |011\rangle\}\right), \tag{42}$$

i.e. the mirror state of equation (41). As noted by Dür *et al* (2000), the $|W\rangle$ state of equation (41) is orthogonal to the $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\{|000\rangle \pm |111\rangle\}$ states. However, this is not surprising, given that the $|\mathrm{GHZ}\rangle$, ($|W\rangle$) states are characterized by $\left|S_Z = \pm\frac{3}{2}\right\rangle$, $\left(\left|S_Z = \pm\frac{1}{2}\right\rangle\right)$, respectively.

A state, more in keeping with the Bell and GHZ states, is obtained on setting $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = \frac{1}{\sqrt{2}}$ and $\nu_{12} = \nu_{13} = \nu_{23} = -\frac{1}{3}$, $\nu_{123} = 0$. We find

$$|W_S\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}(|011\rangle + |101\rangle + |110\rangle) \\ - \\ (|001\rangle + |010\rangle + |001\rangle)\end{pmatrix}. \tag{43}$$

Note that in three $|W\rangle$ cases considered above, exclusive three-particle entanglement (ν_{123}) is always zero.

In practice, it is possible to generate many four component $|W\rangle$−like functions using say $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\nu_{123} = 0$. We find

$$|W\rangle = \frac{1}{\sqrt{1+\nu_{12}^2+\nu_{13}^2+\nu_{23}^2}}\{|+\rangle_1|+\rangle_2|+\rangle_3 + \nu_{12}|-\rangle_1|-\rangle_2|+\rangle_3 + \nu_{13}|-\rangle_1|+\rangle_2|-\rangle_3 \\ + \nu_{23}|+\rangle_1|-\rangle_2|-\rangle_3\} \\ \left(\equiv \frac{1}{\sqrt{1+\nu_{12}^2+\nu_{13}^2+\nu_{23}^2}}\{|000\rangle + \nu_{12}|110\rangle + \nu_{13}|101\rangle + \nu_{23}|011\rangle\}\right) \tag{44}$$

Here the $|W\rangle$−like function of equation (44) is identical to $|\Psi_1\rangle$ of Schlienz and Mahler (1996), provided we set $\nu_{12} = \nu_{13} = \nu_{23} = 1$. (See their equation (2)).

In summary, the $|W\rangle$ functions of Schlienz and Mahler (1996) and Dür *et al* (2000) belong to the image class of joint bi-partite entanglement $\nu_{12}, \nu_{13}, \nu_{23} \neq 0$, with no triple particle entanglement ν_{123}.

9. Concurrences in three-spin 1/2 systems

In sections (9)–(12), we shall assume that the four entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}$ are real. Thus the resulting eight coefficients (*a*–*h*) are also real. Later, in section 13, we shall generalize the entanglement parameters to include a complex term $\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}\mathrm{e}^{\mathrm{i}\phi_{123}}$ i.e. five entanglement parameters. But, as we shall see, there is already sufficient fun, and games, to be had with real coefficients.

In general, it is a relatively easy matter to encode entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}$ into a given state function. However, given the resulting eight coefficients (*a*–*h*), it is quite another matter to retrieve them. Since concurrence can be used to advantage in the two-qubit problem (see section 5), it is obviously of some interest ask whether or not this is also the case in three-spin 1/2 systems. There are some surprises.

Consider first the case where the three spins in question exhibit partial bi-partite ν_{23} and tri-partite $\nu_{123} = (\nu)$ entanglement. Such a state takes the form:

$$|\Psi\rangle = \frac{1}{\eta}\left(\begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix} + \nu_{23}\begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix}\right.$$
$$\left. + \nu_{123}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix}\right)$$
$$\eta = \sqrt{1+\nu_{23}^2+\nu_{123}^2}. \tag{45}$$

For the purposes of this paper, the target wavefunction is defined to be the first term in equation (45), since all the remaining states are derived from this target state, via simple π rotations.

The inverse image $|\Psi_\pi\rangle$ of equation (45) is readily obtained using the π rotation matrix:

$$I_{123}(\pi) = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\otimes\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&1&0\\0&0&0&0&0&1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&1&0&0&0&0\\0&0&-1&0&0&0&0&0\\0&-1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}. \tag{46}$$

We find

$$|\Psi_\pi\rangle = \frac{1}{\eta}\left(\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} + \nu_{23}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix}\right.$$
$$\left. - \nu_{123}\begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix}\right). \tag{47}$$

Thus the scalar product

$$C(\Psi) = \langle\Psi_\pi|\Psi\rangle = \frac{\nu_{123}-\nu_{123}}{\eta^2} \equiv 0. \tag{48}$$

This, at first sight surprising result, is due to a fundamental property of spin 1/2 systems. On inverting the pure state wavefunctions $|\pm\pm\cdots\pm\rangle_n$ we find

$$\begin{aligned} I_n(\pi)|++\cdots+\rangle_n &= (-1)^n|--\cdots-\rangle_n,\\ I_n(\pi)|--\cdots-\rangle_n &= |++\cdots+\rangle_n. \end{aligned} \tag{49}$$

Thus for the GHZ state $|\Psi\rangle_{\mathrm{GHZ}} = \frac{1}{\sqrt{2}}\{|+++\rangle \pm |---\rangle\}$ we obtain

$$C(\Psi) = \langle\Psi_\pi|\Psi\rangle = \tfrac{1}{2}\{\mp\langle---| + \langle+++|\}\bullet\{\pm|+++\rangle + |---\rangle\} = 0, \tag{50}$$

whereas for the four-spin Cat state $|\Psi(\mathrm{Cat})\rangle_4 = \frac{1}{\sqrt{2}}\{|++++\rangle \pm |----\rangle\}$:

$$C(\Psi) = \langle\Psi_\pi|\Psi(\mathrm{Cat})\rangle_4 = \pm 1. \tag{51}$$

It would appear therefore that if n is odd, concurrence cannot be used to recover the entanglement parameter $\nu_{12..n}$. This problem is further examined in appendix C. But, as we shall see below, the situation is not as bleak as it might seem at first sight.

10. Two-particle concurrences in a three-spin 1/2 system

Suppose a new wavefunction is formed where only the particles 2 and 3 are π-reversed. The inverse/image matrix this time is given by

$$I_{23}(\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{52}$$

Thus

$$|\Psi_{\pi(2,3)}\rangle = \frac{1}{\eta}\left(\begin{pmatrix} \alpha_1 \\ -\beta_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_2 \\ \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \beta_3 \\ \alpha_3 \end{pmatrix} + \nu_{23}\begin{pmatrix} \alpha_1 \\ -\beta_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 \\ -\beta_2 \end{pmatrix} \otimes \begin{pmatrix} \alpha_3 \\ -\beta_3 \end{pmatrix} \right.$$
$$\left. + \nu_{123}\begin{pmatrix} \alpha_1 \\ -\beta_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_2 \\ \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \beta_3 \\ \alpha_3 \end{pmatrix}\right). \tag{53}$$

Here the concurrence between $|\Psi_{\pi(2,3)}\rangle$ and $|\Psi\rangle$ is given by

$$C_{23} = \langle \Psi_{\pi(2,3)}|\Psi\rangle = \frac{2\nu_{23}}{1 + \nu_{23}^2 + \nu_{123}^2} = 2(ad + eh - bc - fg). \tag{54}$$

Note that C_{23} depends on both the entanglement parameters ν_{23} and ν_{123}, the latter via the normalization.

This procedure can easily be extended to deal with the general state where all four entanglement parameters $\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}$ are present. We find

$$\begin{aligned} C_{12} &= \langle \Psi_{\pi(12)}|\Psi\rangle = \frac{2(\nu_{12} - \nu_{13}\nu_{23})}{\eta^2} = 2(ag + bh - ce - df), \\ C_{13} &= \langle \Psi_{\pi(13)}|\Psi\rangle = \frac{2(\nu_{13} - \nu_{12}\nu_{23})}{\eta^2} = 2(af + ch - be - dg), \\ C_{23} &= \langle \Psi_{\pi(23)}|\Psi\rangle = \frac{2(\nu_{23} - \nu_{12}\nu_{13})}{\eta^2} = 2(ad + eh - bc - fg). \end{aligned} \tag{55}$$

Finally, if we compute the tri-partite concurrence

$$C_{123} = \langle \Psi_{\pi(123)}|\Psi\rangle = 0 \tag{56}$$

in accord with our earlier comments regarding n-odd spin 1/2 systems. In conclusion, concurrence can be used to project out some details of the bi-partite entanglement factors, albeit with some cross-mixing.

Before leaving this section, it is instructive to consider a few simple cases. Firstly, if no bi-partite entanglements exists $C_{12} = C_{13} = C_{23} \equiv 0$. In general, therefore we have a simple test for the presence or absence of two-particle entanglement. But note that this is also the case when $\nu_{12} = \nu_{13} = \nu_{23} = 1$! Secondly, suppose only one two-particle entanglement ν_{12} exists. We find

$$\begin{aligned} C_{12} &= \frac{2\nu_{12}}{1 + \nu_{12}^2 + \nu_{123}^2}, \\ C_{13} &= C_{23} = 0, \end{aligned} \tag{57}$$

i.e. one equation but two unknowns. Thirdly, if say only $\nu_{23} = 0$ then

$$C_{12} = \frac{2\nu_{12}}{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{123}^2},$$
$$C_{13} = \frac{2\nu_{13}}{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{123}^2}, \tag{58}$$
$$C_{23} = \frac{-2\nu_{12}\nu_{13}}{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{123}^2}.$$

Thus if both ν_{12} and $\nu_{13} < 1$, the two-particle concurrence $C_{23} \propto \nu_{12}\nu_{13}$ is smaller than that of C_{12} and C_{13}. However, equation (55) presents us with a dilemma: three equations and four unknowns. However all is not lost. Other relationships/correlations exist between the eight coefficients.

On expanding the general entangled state we find

$$|\Psi\rangle = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{pmatrix} = \frac{1}{\eta}\begin{pmatrix} \{+\alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\alpha_3\nu_{12} + \beta_1\alpha_2\beta_3\nu_{13} + \alpha_1\beta_2\beta_3\nu_{23} + \beta_1\beta_2\beta_3\nu\} \\ \{-\alpha_1\alpha_2\beta_3 - \beta_1\beta_2\beta_3\nu_{12} + \beta_1\alpha_2\alpha_3\nu_{13} + \alpha_1\beta_2\alpha_3\nu_{23} + \beta_1\beta_2\alpha_3\nu\} \\ \{-\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3\nu_{12} - \beta_1\beta_2\beta_3\nu_{13} + \alpha_1\alpha_2\beta_3\nu_{23} + \beta_1\alpha_2\beta_3\nu\} \\ \{+\alpha_1\beta_2\beta_3 - \beta_1\alpha_2\beta_3\nu_{12} - \beta_1\beta_2\alpha_3\nu_{13} + \alpha_1\alpha_2\alpha_3\nu_{23} + \beta_1\alpha_2\alpha_3\nu\} \\ \{-\beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3\nu_{12} + \alpha_1\alpha_2\beta_3\nu_{13} - \beta_1\beta_2\beta_3\nu_{23} + \alpha_1\beta_2\beta_3\nu\} \\ \{+\beta_1\alpha_2\beta_3 - \alpha_1\beta_2\beta_3\nu_{12} + \alpha_1\alpha_2\alpha_3\nu_{13} - \beta_1\beta_2\alpha_3\nu_{23} + \alpha_1\beta_2\alpha_3\nu\} \\ \{+\beta_1\beta_2\alpha_3 + \alpha_1\alpha_2\alpha_3\nu_{12} - \alpha_1\beta_2\beta_3\nu_{13} - \beta_1\alpha_2\beta_3\nu_{23} + \alpha_1\alpha_2\beta_3\nu\} \\ \{-\beta_1\beta_2\beta_3 - \alpha_1\alpha_2\beta_3\nu_{12} - \alpha_1\beta_2\alpha_3\nu_{13} - \beta_1\alpha_2\alpha_3\nu_{23} + \alpha_1\alpha_2\alpha_3\nu\} \end{pmatrix} \tag{59}$$

$\eta = \sqrt{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{23}^2 + \nu^2}$.

After some manipulation it can be shown that

$$(ah - bg - cf + de) = \left[-2\alpha_1\beta_1(\nu_{23} + \nu_{12}\nu_{13}) + \nu\left(\alpha_1^2 - \beta_1^2\right)\right]/\eta,$$
$$(ah - bg + cf - de) = \left[-2\alpha_2\beta_2(\nu_{13} + \nu_{12}\nu_{23}) + \nu\left(\alpha_2^2 - \beta_2^2\right)\right]/\eta, \tag{60}$$
$$(ah + bg - cf - de) = \left[-2\alpha_3\beta_3(\nu_{12} + \nu_{13}\nu_{23}) + \nu\left(\alpha_3^2 - \beta_3^2\right)\right]/\eta.$$

Note that unlike the concurrences of equation (55) the relationships of equation (60) depend on both bi-partite and tri-partite entanglement factors, and the coefficients (α_1, β_1), etc, but in a relatively simple way. Moreover if we set the target state at $|+++\rangle$ i.e. $\alpha_1 = \alpha_2 = \alpha_3 = 1$ then

$$(ah - bg - cf + ed) = \nu/\eta,$$
$$(ah - bg + cf - ed) = \nu/\eta, \tag{61}$$
$$(ah + bg - cf - ed) = \nu/\eta.$$

In this case, therefore, we have a very simple way of projecting out the three-particle entanglement parameter ν (apart from normalization). Also equation (61) can be used to provide a simple test for determining the presence of pure up or down target wavefunctions. Alternatively, if we set $\alpha_1 = \alpha_3 = 1$ and $\beta_2 = 1$ then

$$(ah - bg - cf + ed) = \nu/\eta,$$
$$(ah - bg + cf - ed) = -\nu/\eta, \tag{62}$$
$$(ah + bg - cf - ed) = \nu/\eta.$$

Clearly, there are many games that can be played. For example, if we set $\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \alpha_3 = \beta_3 = 1/\sqrt{2}$ then

$$(ah - bg - cf + ed) = -(\nu_{23} + \nu_{12}\nu_{13})/\eta,$$
$$(ah - bg + cf - ed) = -(\nu_{13} + \nu_{12}\nu_{23})/\eta, \tag{63}$$
$$(ah + bg - cf - ed) = -(\nu_{12} + \nu_{13}\nu_{23})/\eta.$$

In summary, the bi-partite concurrences of equation (55), plus the new relationships of equation (60) may prove to be useful, either in speeding up the extraction of entanglement factors and/or checking their values.

11. Some simple examples

It is instructive to consider a few simple examples, since one of these provides the basis for the roulette method described in the following section.

Suppose the target function is set at $|++\,+\rangle$, then from equation (59) we have

$$|\Psi\rangle = \frac{1}{\eta}\begin{bmatrix} 1 \\ 0 \\ 0 \\ \nu_{23} \\ 0 \\ \nu_{13} \\ \nu_{12} \\ \nu_{123} \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \\ d \\ 0 \\ f \\ g \\ h \end{bmatrix}. \tag{64}$$

In this case, the entanglement factors can immediately be determined simply by inspection. For example $\nu_{123} = h/a$, $\nu_{12} = g/a$, etc. Similar expressions can be obtained on setting the target function to be $|++\,-\rangle$, $|+-+\rangle$, etc. In all cases there are just three zero entries in $|\Psi\rangle$, whose positions are determined by the form of the target function.

As the next step in increasing complexity, consider the case where the target function is set to $\binom{\alpha_1}{\beta_1} \otimes \binom{1}{0} \otimes \binom{1}{0}$. We find

$$|\Psi\rangle = \frac{1}{\eta}\begin{bmatrix} \alpha_1 \\ \beta_1\nu_{13} \\ \beta_1\nu_{12} \\ \alpha_1\nu_{23} + \beta_1\nu_{123} \\ -\beta_1 \\ \alpha_1\nu_{13} \\ \alpha_1\nu_{12} \\ -\beta_1\nu_{23} + \alpha_1\nu_{123} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}. \tag{65}$$

Thus, the following relationships hold:

$$\begin{aligned} &\alpha_1 = \eta a, \qquad \beta_1 = -\eta e \\ &\nu_{12} = \frac{g}{a} = -\frac{c}{e}, \qquad \nu_{13} = \frac{f}{a} = -\frac{b}{e} \\ &a\nu_{23} - e\nu_{123} = d, \qquad -e\nu_{23} + a\nu_{123} = h. \end{aligned} \tag{66}$$

A simple test for the existence of this type of solution would be to check that both $ac + eg \equiv 0$ and $ab + ef \equiv 0$. Given that this is the case, the four entanglement parameters are given by

$$\begin{aligned} &\nu_{12} = \frac{g}{a}, \qquad \nu_{13} = \frac{f}{a} \\ &\nu_{23} = \frac{ad + eh}{a^2 - e^2}, \qquad \nu_{123} = \frac{ah + de}{a^2 - e^2}. \end{aligned} \tag{67}$$

As the next step, in increasing complexity, consider the case where the target function is set to $\binom{\alpha_1}{\beta_1} \otimes \binom{\alpha_2}{\beta_2} \otimes \binom{1}{0}$. We find

$$|\Psi\rangle = \frac{1}{\eta}\begin{bmatrix} \alpha_1\alpha_2 + \beta_1\beta_2\nu_{12} \\ \beta_1\alpha_2\nu_{13} + \alpha_1\beta_2\nu_{23} + \beta_1\beta_2\nu_{123} \\ -\alpha_1\beta_2 + \beta_1\alpha_2\nu_{12} \\ -\beta_1\beta_2\nu_{13} + \alpha_1\alpha_2\nu_{23} + \beta_1\alpha_2\nu_{123} \\ -\beta_1\alpha_2 + \alpha_1\beta_2\nu_{12} \\ \alpha_1\alpha_2\nu_{13} - \beta_1\beta_2\nu_{23} + \alpha_1\beta_2\nu_{123} \\ \beta_1\beta_2 + \alpha_1\alpha_2\nu_{12} \\ -\alpha_1\beta_2\nu_{13} - \beta_1\alpha_2\nu_{23} + \alpha_1\alpha_2\nu_{123} \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix}. \tag{68}$$

In this situation, one of the new relationships can be used to advantage. From equation (60) we find

$$\begin{aligned} (ah - bg - cf + de) &= \left[-2\alpha_1\beta_1(\nu_{23} + \nu_{12}\nu_{13}) + \nu\left(\alpha_1^2 - \beta_1^2\right)\right]/\eta \\ (ah - bg + cf - de) &= \left[-2\alpha_2\beta_2(\nu_{13} + \nu_{12}\nu_{23}) + \nu\left(\alpha_2^2 - \beta_2^2\right)\right]/\eta \\ (ah + bg - cf - de) &= \nu/\eta. \end{aligned} \tag{69}$$

From the last line of equation (69) therefore, we can essentially project out the tri-partite entanglement factor ν. On combining the latter therefore, with the three concurrences of equation (55), we have four equations and four unknowns.

But for the general target wavefunction, it would appear that there are no simple analytic methods, concurrence or otherwise, which can be used to project out the four entanglement factors available to three-spin 1/2 systems. Thus numerical methods are mandatory. As noted earlier, a numerical solution has already been presented earlier in section 7. But this procedure is tedious when all four entanglement parameters are present. Another, simpler, brute-force method is presented in the following section.

12. The three wheel roulette method

We have already observed that if the target function is a pure state, e.g.$|+++\rangle$, the problem of decoding the entanglement factors is trivial (see equation (64) above). This suggests therefore that if we could rotate the co-ordinate systems of the individual spins, to those where the spins are aligned along their individual z-axes, the problem is solved. However, a little mathematics is required to show that this is indeed possible, starting from the eight coefficients $\{a, b, c, \ldots, h\}$.

Firstly, we take the pure state $|+++\rangle$ as our initial starting point. Secondly, a target state can be generated from the $|+++\rangle$ state using rotations. Explicitly,

$$|\Psi\rangle_T = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \alpha_3 & \beta_3 \\ -\beta_3 & \alpha_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{pmatrix}, \tag{70}$$

where

$$\alpha_1 = \cos\frac{\theta_1}{2}, \qquad \beta_1 = \sin\frac{\theta_1}{2} \text{ etc.} \tag{71}$$

Thirdly, an entangled state is subsequently obtained by applying the various π rotations and adding all the entangled terms together. This can be expressed in the single matrix form:

$$|\Psi\rangle_E = \begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{pmatrix} = \frac{1}{\eta}\begin{pmatrix} 1 & 0 & 0 & \nu_{23} & 0 & \nu_{13} & \nu_{12} & -\nu\\ 0 & 1 & -\nu_{23} & 0 & -\nu_{13} & 0 & \nu & \nu_{12}\\ 0 & -\nu_{23} & 1 & 0 & -\nu_{12} & \nu & 0 & \nu_{13}\\ \nu_{23} & 0 & 0 & 1 & -\nu & -\nu_{12} & -\nu_{13} & 0\\ 0 & -\nu_{13} & -\nu_{12} & \nu & 1 & 0 & 0 & \nu_{23}\\ \nu_{13} & 0 & -\nu & -\nu_{12} & 0 & 1 & -\nu_{23} & 0\\ 0 & -\nu & 0 & -\nu_{13} & 0 & -\nu_{23} & 1 & 0\\ \nu & \nu_{12} & \nu_{13} & 0 & \nu_{23} & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1\alpha_2\alpha_3\\ \alpha_1\alpha_2\beta_3\\ \alpha_1\beta_2\alpha_3\\ \alpha_1\beta_2\beta_3\\ \beta_1\alpha_2\alpha_3\\ \beta_1\alpha_2\beta_3\\ \beta_1\beta_2\alpha_3\\ \beta_1\beta_2\beta_3 \end{pmatrix}. \tag{72}$$

Fourthly, we observe that equations (70) and (72) can be summarized in the matrix form:

$$|\Psi\rangle_E = \begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{pmatrix} = \mathsf{E}(\{\nu\})\mathsf{R}(\{\theta\})\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} = \mathsf{R}(\{\theta\})\mathsf{E}(\{\nu\})\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \tag{73}$$

where (i) $\{\nu\}$ and $\{\theta\}$ are the shorthand notation for the entanglement parameters $\{\nu_{12}, \nu_{13}, \nu_{23}, \nu\}$ and the three rotational angles $\{\theta_1, \theta_2, \theta_3\}$, respectively and (ii) we have used the fact that entanglement $\mathsf{E}(\{\nu\})$ and rotational matrices $\mathsf{R}(\{\theta\})$ commute with each other. Consequently, if we apply the reverse rotational operator to equation (73) we find

$$\mathsf{R}(\{\theta\})^{-1}\begin{pmatrix} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h \end{pmatrix} = \mathsf{E}(\{\nu\})\begin{pmatrix} 1\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix} = \frac{1}{\eta}\begin{pmatrix} 1\\ 0\\ 0\\ \nu_{23}\\ 0\\ \nu_{13}\\ \nu_{12}\\ \nu_{123} \end{pmatrix}. \tag{74}$$

Our task therefore is to discover the three *reverse* rotational matrices:

$$\mathsf{R}(\{\theta\})^{-1} = \begin{pmatrix} \alpha_1 & -\beta_1\\ \beta_1 & \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \alpha_2 & -\beta_2\\ \beta_2 & \alpha_2 \end{pmatrix} \otimes \begin{pmatrix} \alpha_3 & -\beta_3\\ \beta_3 & \alpha_3 \end{pmatrix}, \tag{75}$$

which satisfy equation (74). Note that this method works because the entanglement and rotational matrices $\mathsf{E}(\{\nu\})$ and $\mathsf{R}(\{\theta\})$ commute. This is also the case for the four-spin 1/2 problem, characterized by 16×16 matrices and a maximum of 11 real entanglement parameters. But this is not surprising, given that the entanglement factors are invariant under local rotations.

In practice, the three wheel roulette method requires an algorithm which cycles over the three angles $\{\theta_1, \theta_2, \theta_3\}$, until three zeroes are found on the right-hand side of equation (74), in the places specified. Since this involves spinning the angles $\theta_1, \theta_2, \theta_3$, repeatedly, it is natural to call this procedure: the roulette method. In practice, the method works well if integer values are used for the angles α_1, β_1, *etc*, used to generate the target wavefunction: $\binom{\alpha_1}{\beta_1} \otimes \binom{\alpha_2}{\beta_2} \otimes \binom{\alpha_3}{\beta_3}$. But if the encoder chooses to use transcendental numbers for $\theta_1, \theta_2, \theta_3$, it soon becomes very tedious. Indeed, the procedure could then be described as the Devils' roulette, requiring ever finer angular steps. But, at worst, it could be used to provide approximate values of $\{\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}\}$, which can then be used as I/P parameters to say a FindRoots routine, as detailed above in section 7.

13. Classification of image-entangled states

From an examination of the general entangled wavefunction of equation (36) it is evident that the three- and two-particle *image* wavefunctions are all orthogonal to each other, and individually to the primary *object* wavefunction. Therefore, provided the entanglement factors are real, we have a natural classification for entangled image states: one exclusive three-particle ν_{123} and three two-particle $\nu_{12}, \nu_{13}, \nu_{23}$ parameters. But this does not rule out the existence of sub-classes. For example, a mixture of bi-partite $\nu_{12}, \nu_{13}, \nu_{23} \neq 0$ entanglement can be used to generate the $|W\rangle$ functions of Dür *et al* (2000) (see section 9). In all, for the three-spin 1/2 problem, there are a possible eight sub-classes. These are (i) simple product states($\nu_{12} = \nu_{13} = \nu_{23} = \nu_{123} = 0$), (ii) the three bi-partite sub-classes $\{(\nu_{12}), (\nu_{12}, \nu_{13}), (\nu_{12}, \nu_{13}, \nu_{23})\}$, where it is understood that the sub-class (ν_{12}, ν_{13}) includes (ν_{12}, ν_{23}) and (ν_{13}, ν_{23}), and (iii) joint bi-partite and tri-partite states: $\{(\nu_{123}), (\nu_{12}, \nu_{123}), (\nu_{12}, \nu_{13}, \nu_{123}), (\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123})\}$. These can be used to classify all possible states from fully separable wavefunctions, to W-functions, partially and fully entangled GHZ states, etc.

We now make contact with the work of Acin *et al* (2000, 2001). Using a generalization of the Schmidt decomposition, the former have shown that the three-qubit problem can be reduced to five (as opposed to eight) local basis states (see Acin *et al* 2000, equation (2), where three differing basis states are listed). This conclusion finds a natural resonance with equation (36), which also employs five orthogonal functions. However, Acin *et al* have shown that in addition to the reduction to five basis sets, the general entangled three-qubit system requires five entanglement parameters. This therefore appears to clash with the image approach where so far just four entanglement factors $(\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123})$ have been employed. However, in appendix D it is shown that, in general, five parameters are indeed required: $(\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}, \phi_{123})$. Phase shifts associated with bi-particle entanglement can be absorbed into the basis states, but complete phase absorption turns out not to be possible in the presence of both bi-partite and triple-partite entanglements. Thus, the minimalistic general entangled three-qubit pure state takes the form:

$$|\Psi\rangle = \frac{1}{\eta}[|+++\rangle + \nu_{12}|--+\rangle + \nu_{13}|-+-\rangle + \nu_{23}|+--\rangle + \nu_{123}\,\mathrm{e}^{\mathrm{i}\phi_{123}}|---\rangle], \tag{76}$$

where (i) $|\nu_{12}|^2 + |\nu_{13}|^2 + |\nu_{23}|^2 + |\nu_{123}|^2 \leqslant 1$ and (ii) $0 \leqslant \phi_{123} \leqslant 2\pi$. This simple state, can be used to illustrate, in a very transparent way, all of the eight entangled classes W GHZ, etc, that arise in three-spin 1/2 entangled systems. Note also that (i) the meaning of the factors $(\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123})$ is immediately apparent and (ii) the phase shift ϕ_{123} can be interpreted as an extra rotation of the triple-inverse image about the z-axis. A schematic diagram of the general entangled state of equation (76) can be seen in figure 3.

Finally, it should be noted that $|\Psi\rangle$ of equation (76) is similar to that proposed by Acin *et al* (2000, 2001), namely

$$|\Psi_A\rangle = [\lambda_0|+++\rangle + \lambda_1\,\mathrm{e}^{\mathrm{i}\phi}|-++\rangle + \lambda_2|-+-\rangle + \lambda_3|--+\rangle + \lambda_4|---\rangle], \tag{77}$$

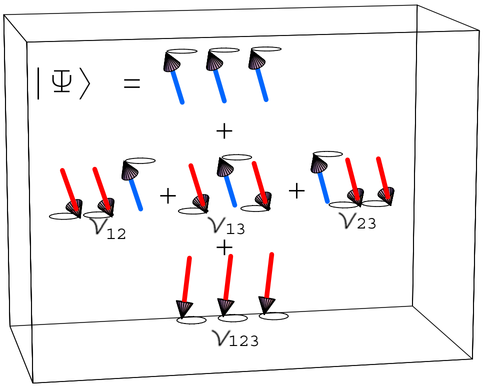

Figure 3. A schematic representation of an entangled three-spin 1/2 system. All three bi-partite entangled functions are phase locked with $\theta \equiv 0°$ or 180° and $\phi \equiv 0°$ to the target state $|++\,+\rangle$. Note that the target state spins are all parallel. This is not strictly necessary, but the phases of the individual spins in both the target and bi-partite entangled functions should be 100% correlated. The exception is the tri-partite entangled function, which is related to the target state by an extra phase rotation of ϕ_{123}.

i.e. five basis states and five entanglement parameters, as mentioned earlier. Here Acin *et al* have chosen, arbitrarily, to assign the phase shift ϕ to second term. This is in marked contrast to the image approach, where there is a good reason for assigning the phase factor to the $|---\rangle$ state (see appendix D). However, note that if $\lambda_2 = \lambda_3 = \lambda_4 = 0$, equation (77) can be fully factorized. Similar difficulties also exist for the other two sets proposed by Acin *et al*.

These problems however, can be resolved, and the two approaches reconciled, if it is assumed that in contrast to $\nu_{12} - \nu_{123}$ parameters, the coefficients $\lambda_0 - \lambda_4$ are not independent of each other. To check this point, we have re-cast the image state of equation (76) into the form advocated by Acin *et al* i.e. by first performing a unitary transformation on the first qubit (with the requirement that $|T_0'| = 0$), followed by singular value decomposition on the remaining two qubits. In general, the relationship between the coefficients $\lambda_0 - \lambda_4$ and $\nu_{12} - \nu_{123}$ is complex. For example, on setting the determinant $|T_0'| = 0$ we find

$$\nu_{23}\cos^2\theta + \nu_{123}\sin\theta\cos\theta - \nu_{12}\nu_{13}\sin^2\theta = 0, \tag{78}$$

where θ defines the unitary transformation on the first qubit. In view of this complexity therefore, we have used a numerical approach, to obtain the complementary form equation (77) given equation (76). In general, it is found that the coefficients $\lambda_0 - \lambda_4$ and $\nu_{12} - \nu_{123}$ differ markedly. But when the partial entropies of entanglement are computed, using the methods outlined by Ekert and Knight (1995), but extended to the three-spin 1/2 system, they are found to be identical. In passing, we note that the phase factor ϕ_{123} plays no role in determining the partial entropies of the three spins.

In summary therefore, the number of entanglement parameters $\nu_{12}, \nu_{13}, \ldots,$ and concomitant sub-classes, will increase rapidly with increasing numbers of spins. If all the parameters are real, we find the statistics summarized in table 1. In general, it is easily shown, using table 1, or otherwise, that the number of real parameters is given by $N_\nu = 2^n - (n+1)$.

If we now include phases we find the results summarized in table 2. In general, we can only absorb n entanglement phases for n spins (see appendix D).

We are now in a position to make contact with the work of Linden and Popescu (1997). First, we note that the number of entanglement parameters given in table 2:

Table 1. Numbers of entanglement parameters and sub-classes as function of N_ν for spin 1/2 systems, for real entanglement factors.

n	ν_{ij}	ν_{ijk}	ν_{ijkl}	ν_{ijklm}	ν_{ijklmn}	Total N_ν	No. sub-classes
2	1					1	2
3	3	1				4	8
4	6	4	1			11	60
5	10	1	5	1		26	1127
6	15	20	15	6	1	57	58 858

Table 2. Total numbers of entanglement parameters (real plus phases) for spin 1/2 systems with $n \leqslant 6$. In general, $N_{\nu\phi} = 2(N_\nu) - n = 2^{n+1} - (3n+2)$.

n	ν_{ij}	$\nu_{ijk}(\phi_{ijk})$	$\nu_{ijkl}(\phi_{ijkl})$	$\nu_{ijklm}(\phi_{ijklm})$	$\nu_{ijklmn}(\phi_{ijklmn})$	Total $N_\nu(N_\phi)$	Total$N_{\nu\phi}$
2	1					1	1
3	3(0)	1(1)				4(1)	5
4	6(6)	4(0)	1(1)			11(7)	18
5	10(10)	10(10)	5(0)	1(1)		26(21)	47
6	15(15)	20(20)	15(15)	6(0)	1(1)	57(51)	108

$N_{\nu\phi} = 2^{n+1} - (3n+2)$ is similar to that given by Linden and Popescu: $N \geqslant 2^{n+1} - (3n+1)$, obtained by simple counting. However, in this work, we have been at pains to drop the phase factor associated with the target state. For example, consider the general state belonging to the unitary group $U(2) \otimes U(2)$:

$$\begin{aligned}|\Psi\rangle &= a\,\mathrm{e}^{\mathrm{i}\phi_{++}}|{+}{+}\rangle + b\,\mathrm{e}^{\mathrm{i}\phi_{+-}}|{+}{-}\rangle + c\,\mathrm{e}^{\mathrm{i}\phi_{-+}}|{-}{+}\rangle + d\,\mathrm{e}^{\mathrm{i}\phi_{--}}|{-}{-}\rangle \\ &= \mathrm{e}^{\mathrm{i}\phi_{++}}(a|{+}{+}\rangle + b\,\mathrm{e}^{\mathrm{i}\phi'_{+-}}|{+}{-}\rangle + c\,\mathrm{e}^{\mathrm{i}\phi'_{-+}}|{-}{+}\rangle + d\,\mathrm{e}^{\mathrm{i}\phi'_{--}}|{-}{-}\rangle).\end{aligned} \tag{79}$$

Here $2^{n+1}(n=2) = 8$ are free parameters. However, in this paper, we have simply dropped the phase factor $\mathrm{e}^{\mathrm{i}\phi_{++}}$, as being unimportant:

$$|\Psi\rangle \Longrightarrow (a|{+}{+}\rangle + b\,\mathrm{e}^{\mathrm{i}\phi'_{+-}}|{+}{-}\rangle + c\,\mathrm{e}^{\mathrm{i}\phi'_{-+}}|{-}{+}\rangle + d\,\mathrm{e}^{\mathrm{i}\phi'_{--}}|{-}{-}\rangle). \tag{80}$$

So the number of available parameters is inevitably reduced to $2^{n+1} - 1$, for all n. Second, we note that the Linden and Popescu have suggested that there may be more entanglement parameters, other than those obtained by simple counting (hence the lower bound). No examples were given, but if we insist that the basis set is fixed, with say all phases set equal to zero, then, by definition, the entanglement-phase parameters $\phi_{ij,\dots,k}$ cannot be absorbed into the basis set and must be counted as free parameters. Consequently, the number of entanglement parameters rises to a maximum of $N_{\max} = 2^{n+1} - 2(n+1)$.

From tables 1 and 2, it is immediately apparent that the number of entanglement parameters rapidly exceeds the number of local parameters, as the number of spins n is increased (see Linden and Popescu 1997). For the general six-spin 1/2 configuration, 57 real entanglement parameters are possible and an astonishing 58 858 sub-classes: more if we allow $\phi_{123} \neq 0$, etc. In practice, this would pose considerable problems for a decoder, say Bob, trying to retrieve the entanglement factors built by Alice into a given $|\Psi\rangle$. Perhaps such problems could be relieved by the sender, say Alice, providing some details of the wavefunction by an alternative secure route. However, the situation is trivial if only exclusive n-particle entangled Cat-states are involved.

Finally, it might be helpful to some readers to give a simple allegory of the entanglement problem. In figure 1, we have portrayed the partially entangled two-spin 1/2 system as one with a linear superposition of an object wavefunction with its inverse image: generalized Bell entanglement. However, in multi-spin 1/2 systems the situation is much more complex. Consider Alice as a representation of a four-coupled-spin system i.e. her head, torso, arms and legs. We can mimic an entangled state as one where Alice is entangled with her own image in the looking glass. On increasing the degree of pure four-spin entanglement, the image of Alice in the mirror will increase from nothing to a complete image. But in quantum mechanics, Alice's image is as important as Alice herself. So as the image becomes stronger, Alice herself must decrease to preserve normalization. If we now allow partial-body entanglement, the situation becomes even more curious. For example, Alice could be partially entangled with only her torso, arms and legs. Thus the image of Alice in the mirror would be complete except for her head. In general therefore, with differing entanglement factors, parts of Alice could appear stronger in the flesh than those in the mirror, and vice versa. In addition, we note that perhaps a more accurate description of the mirror can be achieved using the analogy of a pin-hole camera. The point of inversion is then the pin hole itself. So if Alice is standing up, her image through the pin hole will be upside down. Finally, the allegory takes an even stranger twist if allow Alice to be fully entangled with herself, but with a slowly time-dependent phase factor $\phi_{1-4}(t)$. While Alice is standing still, her image in the mirror will slowly rotate.

14. Conclusions

In this paper, a different approach to entanglement in spin 1/2 systems has been presented and discussed. Central to our arguments is the method of images, which states that any entangled state can be interpreted as a sum of a separable object wavefunction plus a portion of its own inverse image. In particular, a generalized Bell state has been defined, which makes use of the well-known $SU(2)$ rotation operators $\mathcal{D}^{1/2}(\alpha'\beta'\gamma')$. This procedure can be generalized to three or more particles.

Special attention has been paid to the image entanglement of three-spin 1/2 particles. It has been shown that this simple system exhibits complex behaviour, involving either exclusive three-particle ν_{123} entanglement, or two-particle entanglement $\nu_{12}, \nu_{13}, \nu_{23}$, and/or mixtures of all four. In all there are eight distinct image sub-classes, for *real* entanglement parameters. Clearly, entanglement cannot be described and/or quantified in terms of a single parameter. (See also Dür *et al* 2000, Acin *et al* 2000, 2001, Miyake 2003, Sugita 2008.)

In addition, it has been demonstrated that there is considerable scope for encoding numbers in entangled pure states, both from quantum and classical points of view. In general, encrypting numbers into a partially entangled three-spin 1/2 pure state, is a relatively straight forward matter. One simply selects (i) the starting angles $\theta_1, \theta_2, \theta_3$, used to generate, the target wavefunction: $(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ and (ii) the entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}, \nu$ (say the encrypted numbers). These are subsequently used to generate the eight coefficients $(a, b, \ldots, h)$. However, computing the inverse, to unravel the above, is quite another matter. Numerical methods appear to be mandatory. Two methods have been described. The first involves solving a set of four coupled equations, to extract the entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}, \nu$. However, this method requires the decoder to select approximate starting values for the four entanglement factors $\nu_{12}, \nu_{13}, \nu_{23}, \nu$. Many attempts might therefore be necessary. A second, brute force method, the roulette method, has also been described. In practice, this procedure works well if integer angles $\theta_1, \theta_2, \theta_3$ are used in the encoding process. But, if the sender chooses to use transcendental values for $\theta_1, \theta_2, \theta_3$, the method rapidly turns into the Devil's roulette, requiring finer and finer angular steps.

Finally, we remark that although we have been primarily concerned with the $n = 3$ spin 1/2 system, many of the procedures outlined in this paper will find applications in entangled spin 1/2 states with $n > 3$. For example, if $n = 4$, 5 and 6 no-less than 11, 26 and 57 differing but *real* entanglement factors, respectively, could be encoded into a single pure state. Indeed, the development of new decoding algorithms may well prove to be essential, even for a simple $n = 4$ spin 1/2 system.

Acknowledgment

The author wishes to thank the referees for their helpful comments.

Appendix A. Two-spin 1/2 entanglement algorithm with complex numbers

Starting from the spin-up state $\left|\frac{1}{2}\right\rangle$, we generate a general object wavefunction:

$$|\Psi\rangle_O = \mathcal{D}^{1/2}(\alpha'\beta'\gamma')\left|\tfrac{1}{2}\right\rangle, \tag{A.1}$$

where the rotation matrix for an *active* rotation is given by

$$\mathcal{D}^{1/2}(\alpha'\beta'\gamma') = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}(\alpha'+\gamma')/2}\cos\left(\frac{\beta'}{2}\right) & -\mathrm{e}^{-\mathrm{i}(\alpha'-\gamma')/2}\sin\left(\frac{\beta'}{2}\right) \\ +\mathrm{e}^{\mathrm{i}(\alpha'-\gamma')/2}\sin\left(\frac{\beta'}{2}\right) & \mathrm{e}^{\mathrm{i}(\alpha'+\gamma')/2}\cos\left(\frac{\beta'}{2}\right) \end{bmatrix}. \tag{A.2}$$

Thus

$$\begin{aligned} |\Psi\rangle_O &= \mathrm{e}^{-\mathrm{i}\gamma'/2}\left(\mathrm{e}^{-\mathrm{i}\alpha'/2}\cos\left(\frac{\beta'}{2}\right)\left|\frac{1}{2}\right\rangle + \mathrm{e}^{+\mathrm{i}\alpha'/2}\sin\left(\frac{\beta'}{2}\right)\left|-\frac{1}{2}\right\rangle\right) \\ &= \left(\mathrm{e}^{-\mathrm{i}\alpha'/2}\alpha\left|\frac{1}{2}\right\rangle + \mathrm{e}^{+\mathrm{i}\alpha'/2}\beta\left|-\frac{1}{2}\right\rangle\right), \end{aligned} \tag{A.3}$$

where (i)

$$\alpha = \cos\left(\frac{\beta'}{2}\right), \qquad \beta = \sin\left(\frac{\beta'}{2}\right), \tag{A.4}$$

i.e. real quantities and (ii) we have dropped the overall phase factor $\mathrm{e}^{-\mathrm{i}\gamma'/2}$. Consequently, for the complex version of the entanglement algorithm we write

$$|\Psi\rangle_O = \frac{1}{\sqrt{1+\nu^2}}((\alpha_1\,\mathrm{e}^{-\mathrm{i}\phi(1)}|+\rangle_1 + \beta_1\,\mathrm{e}^{\mathrm{i}\phi(1)}|-\rangle_1)\otimes(\alpha_2\,\mathrm{e}^{-\mathrm{i}\phi(2)}|+\rangle_2 + \beta_2\,\mathrm{e}^{\mathrm{i}\phi(2)}|-\rangle_2))$$

and

$$|\Psi\rangle_I = \frac{\nu}{\sqrt{1+\nu^2}}((\beta_1\,\mathrm{e}^{-\mathrm{i}\phi(1)}|+\rangle_1 - \alpha_1\,\mathrm{e}^{\mathrm{i}\phi(1)}|-\rangle_1)\otimes(\beta_2\,\mathrm{e}^{-\mathrm{i}\phi(2)}|+\rangle_2 - \alpha_2\,\mathrm{e}^{\mathrm{i}\phi(2)}|-\rangle_2)), \tag{A.5}$$

where (i) $\alpha_1, \beta_1, \alpha_2$ and β_2 are real quantities, and (ii) $\phi(1) = \alpha'(1)/2$ and $\phi(2) = \alpha'(2)/2$. Thus

$$\langle\Psi_O|\Psi_I\rangle = (\alpha_1\beta_1 - \beta_1\alpha_1)(\alpha_2\beta_2 - \beta_2\alpha_2) = 0 \tag{A.6}$$

as required. Note also that the phases could be absorbed into the spin-up and spin-down wavefunctions, leaving just real quantities $\alpha_1, \beta_1, \alpha_2, \beta_2$, as asserted in the text (see also Peres 1995a).

Finally, we observe that it would be possible to attach a further phase to the image function $|\Psi\rangle_I$, in effect, making the entanglement parameter ν complex. For example consider the Bell state:

$$\begin{aligned} |\Psi\rangle &= \left(\left|\tfrac{1}{2}\right\rangle_1\left|\tfrac{1}{2}\right\rangle_2 + \nu\,\mathrm{e}^{+\mathrm{i}\varphi}\left|-\tfrac{1}{2}\right\rangle_1\left|-\tfrac{1}{2}\right\rangle_2\right)\Big/\sqrt{1+\nu^2} \\ &= \mathrm{e}^{+\mathrm{i}\varphi/2}\left(\mathrm{e}^{-\mathrm{i}\varphi/2}\left|\tfrac{1}{2}\right\rangle_1\left|\tfrac{1}{2}\right\rangle_2 + \nu\mathrm{e}^{+\mathrm{i}\varphi/2}\left|-\tfrac{1}{2}\right\rangle_1\left|-\tfrac{1}{2}\right\rangle_2\right)\sqrt{1+\nu^2}. \end{aligned} \tag{A.7}$$

On comparing equations (A.3) and (A.7), we conclude that the phase φ could be absorbed into the either of the basis states $\left|\pm\frac{1}{2}\right\rangle_1 \left|\pm\frac{1}{2}\right\rangle_2$, leaving the entanglement factor ν real, in accord with the Schmidt decomposition. This trick however, does not work, in general, for the three-spin 1/2 problem (see appendices B and D).

Appendix B. Three-spin 1/2 entanglement algorithm with complex numbers

Consider the case where ν_{23} and $\nu_{123}(=\nu)$ are non-zero. We find

$$
\begin{aligned}
|\Psi\rangle_O &= \frac{1}{\eta} \\
&\times \begin{Bmatrix}
\alpha_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_+(3))}|+\rangle_1|+\rangle_2|+\rangle_3 + \alpha_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_-(3))}|+\rangle_1|+\rangle_2|-\rangle_3 + \\
\alpha_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_+(3))}|+\rangle_1|-\rangle_2|+\rangle_3 + \alpha_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_-(3))}|+\rangle_1|-\rangle_2|-\rangle_3 + \\
\beta_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_+(3))}|-\rangle_1|+\rangle_2|+\rangle_3 + \beta_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_-(3))}|-\rangle_1|+\rangle_2|-\rangle_3 + \\
\beta_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_+(3))}|-\rangle_1|-\rangle_2|+\rangle_3 + \beta_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_-(3))}|-\rangle_1|-\rangle_2|-\rangle_3
\end{Bmatrix}, \\
|\Psi\rangle_I &= \frac{\nu_{23}}{\eta} \\
&\times \begin{Bmatrix}
\alpha_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_+(3))}|+\rangle_1|+\rangle_2|+\rangle_3 - \alpha_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_-(3))}|+\rangle_1|+\rangle_2|-\rangle_3 + \\
-\alpha_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_+(3))}|+\rangle_1|-\rangle_2|+\rangle_3 + \alpha_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_-(3))}|+\rangle_1|-\rangle_2|-\rangle_3 + \\
-\beta_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_+(3))}|-\rangle_1|+\rangle_2|+\rangle_3 + \beta_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_-(3))}|-\rangle_1|+\rangle_2|-\rangle_3 + \\
\beta_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_+(3))}|-\rangle_1|-\rangle_2|+\rangle_3 - \beta_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_-(3))}|-\rangle_1|-\rangle_2|-\rangle_3
\end{Bmatrix} \\
&+\frac{\nu}{\eta} \begin{Bmatrix}
\beta_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_+(3))}|+\rangle_1|+\rangle_2|+\rangle_3 - \beta_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_+(2)+\phi_-(3))}|+\rangle_1|+\rangle_2|-\rangle_3 + \\
-\beta_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_+(3))}|+\rangle_1|-\rangle_2|+\rangle_3 + \beta_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_+(1)+\phi_-(2)+\phi_-(3))}|+\rangle_1|-\rangle_2|-\rangle_3 + \\
-\alpha_1\beta_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_+(3))}|-\rangle_1|+\rangle_2|+\rangle_3 + \alpha_1\beta_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_+(2)+\phi_-(3))}|-\rangle_1|+\rangle_2|-\rangle_3 + \\
\alpha_1\alpha_2\beta_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_+(3))}|-\rangle_1|-\rangle_2|+\rangle_3 - \alpha_1\alpha_2\alpha_3\, \mathrm{e}^{\mathrm{i}(\phi_-(1)+\phi_-(2)+\phi_-(3))}|-\rangle_1|-\rangle_2|-\rangle_3
\end{Bmatrix} \\
\eta^2 &= \left(1+\nu_{23}^2+\nu^2\right).
\end{aligned} \tag{B.1}
$$

Given that the initial state is of the form:

$$
\begin{aligned}
|\Psi\rangle = {}& a\,\mathrm{e}^{\mathrm{i}\phi_{+++}}|+\rangle_1|+\rangle_2|+\rangle_3 + b\,\mathrm{e}^{\mathrm{i}\phi_{++-}}|+\rangle_1|+\rangle_1|-\rangle_3 + c\,\mathrm{e}^{\mathrm{i}\phi_{+-+}}|+\rangle_1|-\rangle_2|+\rangle_3 + d\,\mathrm{e}^{\mathrm{i}\phi_{+--}}|+\rangle_1|-\rangle_2|-\rangle_3 \\
& + e\,\mathrm{e}^{\mathrm{i}\phi_{-++}}|-\rangle_1|+\rangle_2|+\rangle_3 + f\,\mathrm{e}^{\mathrm{i}\phi_{-+-}}|-\rangle_1|+\rangle_1|-\rangle_3 + g\,\mathrm{e}^{\mathrm{i}\phi_{--+}}|-\rangle_1|-\rangle_2|+\rangle_3 \\
& + h\,\mathrm{e}^{\mathrm{i}\phi_{---}}|-\rangle_1|-\rangle_2|-\rangle_3.
\end{aligned} \tag{B.2}
$$

We find that

$$
\begin{aligned}
\phi_{+++} &= \phi_+(1)+\phi_+(2)+\phi_+(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(-\alpha_1'-\alpha_2'-\alpha_3'), \\
\phi_{++-} &= \phi_+(1)+\phi_+(2)+\phi_-(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(-\alpha_1'-\alpha_2'+\alpha_3'), \\
\phi_{+-+} &= \phi_+(1)+\phi_-(2)+\phi_+(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(-\alpha_1'+\alpha_2'-\alpha_3'), \\
\phi_{+--} &= \phi_+(1)+\phi_-(2)+\phi_-(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(-\alpha_1'+\alpha_2'+\alpha_3'), \\
\phi_{-++} &= \phi_-(1)+\phi_+(2)+\phi_+(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(+\alpha_1'-\alpha_2'-\alpha_3'), \\
\phi_{-+-} &= \phi_-(1)+\phi_+(2)+\phi_-(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(+\alpha_1'-\alpha_2'+\alpha_3'), \\
\phi_{--+} &= \phi_-(1)+\phi_-(2)+\phi_+(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(+\alpha_1'+\alpha_2'-\alpha_3'), \\
\phi_{---} &= \phi_-(1)+\phi_-(2)+\phi_-(3) = -\tfrac{1}{2}(\gamma_1'+\gamma_2'+\gamma_3') + \tfrac{1}{2}(+\alpha_1'+\alpha_2'+\alpha_3').
\end{aligned} \tag{B.3}
$$

Again, we can never hope to recover the overall phase $(\gamma_1' + \gamma_2' + \gamma_3')$ which is common to all coefficients. However, given the eight complex coefficients (*a*–*h*), we can recover the individual phases $(\alpha_1', \alpha_2', \alpha_3')$. For example we find

$$\begin{aligned}
\phi_{+++} - \phi_{++-} &= -\alpha_3', \\
\phi_{+-+} - \phi_{+--} &= -\alpha_3', \\
\phi_{-++} - \phi_{-+-} &= -\alpha_3', \\
\phi_{--+} - \phi_{---} &= -\alpha_3'.
\end{aligned} \tag{B.4}$$

So not only can we recover the $(\alpha_1', \alpha_2', \alpha_3')$, we can also check for self-consistency.

Appendix C. Conjunct entangled states

Consider the general entangled image state:

$$|\Psi\rangle = \frac{1}{\eta}\begin{pmatrix} \begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix} \\ +\nu_{12}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix} + \nu_{13}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \\ +\nu_{23}\begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \\ +\nu_{123}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \end{pmatrix} \tag{C.1}$$

$$\eta = \sqrt{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{23}^2 + \nu_{123}^2}.$$

We define the conjunct state to equation (C.1) as

$$|\Psi_C\rangle = \frac{1}{\eta}\begin{pmatrix} \begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix} \\ +\nu_{12}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\alpha_3\\-\beta_3\end{pmatrix} + \nu_{13}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\alpha_2\\-\beta_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \\ +\nu_{23}\begin{pmatrix}\alpha_1\\-\beta_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \\ -\nu_{123}\begin{pmatrix}\beta_1\\\alpha_1\end{pmatrix}\otimes\begin{pmatrix}\beta_2\\\alpha_2\end{pmatrix}\otimes\begin{pmatrix}\beta_3\\\alpha_3\end{pmatrix} \end{pmatrix} \tag{C.2}$$

$$\eta = \sqrt{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{23}^2 + \nu_{123}^2}.$$

Note that $|\Psi\rangle$ and $|\Psi_C\rangle$ are identical except for the change of sign of the tri-partite entanglement parameter ν_{123}.

If we now form the concurrence $C_{123}^c = \langle\Psi_{\pi(123)}|\Psi_C\rangle$ we find

$$C_{123}^c = \langle\Psi_{\pi(123)}|\Psi_C\rangle = \frac{2\nu_{123}}{\sqrt{1 + \nu_{12}^2 + \nu_{13}^2 + \nu_{23}^2 + \nu_{123}^2}}. \tag{C.3}$$

Thus, it is possible to directly project out the entanglement factor ν_{123}, but only if both the entangled $|\Psi\rangle$ and its conjunct state $|\Psi_C\rangle$ are available. This argument can be extended to higher numbers of odd numbered spins.

Appendix D. Complex entanglement factors

In appendix A, it was shown that if a phase was attached to the bi-partite entanglement factor ν, it could be absorbed into the basis states, rendering ν positive semi-definite in accord with the Schmidt decomposition. It is therefore reasonable to ask whether or not similar absorption can be exploited in three-spin 1/2 systems.

Consider the general image-entangled state:

$$|\Psi\rangle = |+++\rangle + \nu_{12}\,\mathrm{e}^{\mathrm{i}\varphi_{12}}|--+\rangle + \nu_{13}\,\mathrm{e}^{\mathrm{i}\varphi_{13}}|-+-\rangle + \nu_{23}\,\mathrm{e}^{\mathrm{i}\varphi_{13}}|+--\rangle + \nu_{123}\,\mathrm{e}^{\mathrm{i}\varphi_{123}}|---\rangle. \tag{D.1}$$

Here we have used $|+++\rangle$ as the target (object) function.

Next, imagine that we wish to associate a given phase with each qubit i.e.

$$|\psi(+)\rangle_1 = |+\rangle_1\,\mathrm{e}^{-\mathrm{i}\phi_1}, \qquad |\psi(-)\rangle_1 = |-\rangle_1\,\mathrm{e}^{+\mathrm{i}\phi_1} \quad \text{etc} \tag{D.2}$$

see, for example equation (A.3). The equivalent of equation (D.1) therefore takes the form:

$$\begin{aligned} |\Psi\rangle = {}& |+++\rangle\,\mathrm{e}^{-\mathrm{i}(\phi_1+\phi_2+\phi_3)} + \nu_{12}\,\mathrm{e}^{\mathrm{i}(\phi_1+\phi_2-\phi_3)}|--+\rangle + \nu_{13}\,\mathrm{e}^{\mathrm{i}(\phi_1-\phi_2+\phi_3)}|-+-\rangle \\ & + \nu_{23}\,\mathrm{e}^{\mathrm{i}(-\phi_1+\phi_2+\phi_3)}|+--\rangle + \nu_{123}\,\mathrm{e}^{+\mathrm{i}(\phi_1+\phi_2+\phi_3)}|---\rangle. \end{aligned} \tag{D.3}$$

Or alternatively

$$\begin{aligned} \mathrm{e}^{+\mathrm{i}(\phi_1+\phi_2+\phi_3)}|\Psi\rangle = {}& |+++\rangle + \nu_{12}\,\mathrm{e}^{\mathrm{i}(2\phi_1+2\phi_2)}|--+\rangle + \nu_{13}\,\mathrm{e}^{\mathrm{i}(2\phi_1+2\phi_3)}|-+-\rangle \\ & + \nu_{23}\,\mathrm{e}^{\mathrm{i}(2\phi_2+2\phi_3)}|+--\rangle + \nu_{123}\,\mathrm{e}^{+\mathrm{i}2(\phi_1+\phi_2+\phi_3)}|---\rangle. \end{aligned} \tag{D.4}$$

If we ignore the triple-entanglement term, a solution can be obtained for ϕ_1, ϕ_2 and ϕ_3. Explicitly,

$$\begin{aligned} \phi_1 &= \tfrac{1}{4}(\varphi_{12} + \varphi_{13} - \varphi_{23}), \\ \phi_2 &= \tfrac{1}{4}(\varphi_{12} - \varphi_{13} + \varphi_{23}), \\ \phi_3 &= \tfrac{1}{4}(-\varphi_{12} + \varphi_{13} + \varphi_{23}). \end{aligned} \tag{D.5}$$

We conclude therefore that for bi-partite entanglement, associated phases can be incorporated into the basis sets. Consequently, in minimalistic form:

$$|\Psi\rangle = |+++\rangle + \nu_{12}|--+\rangle + \nu_{13}|-+-\rangle + \nu_{23}|+--\rangle + \nu_{123}\,\mathrm{e}^{\mathrm{i}\varphi_{123}}|---\rangle, \tag{D.6}$$

i.e. a total of five entanglement parameters ($\nu_{12}, \nu_{13}, \nu_{23}, \nu_{123}$ and φ_{123}). This number is in agreement with Acin *et al* (2000, 2001) who reached the same conclusion, via an entirely different route. Note that (i) equation (1) of Acin *et al* (2001) is similar too, but not identical with that of equation (D.6), (ii) no matter what choice of parameters is made equation (D.6) cannot be factorized (some two-particle or three-particle entanglement always exists). However this is not the case for equation (10) of Acin *et al* (2000) and equation (1) of Acin *et al* (2001).

Finally, we note the following. First, equation (D.6) is not unique, in that we could have used a different target state. However, the conclusions are independent of the target state. Second, on generalizing the above discussion to *n*-spins, it can be shown that it is only ever possible to absorb *n* entanglement phases into the basis states, in accord with table 2. Third, for two particles (regardless of spin) we are guaranteed by the singular value decomposition theorem (e.g. Paškauskas and You 2001) that any entanglement phases can be absorbed into the basis set. But this is not the case for three or more particles.

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